UDC 538.4

ON A MODEL OF CONTINUOUS MEDIUM WITH ELECTROMAGNETIC EFFECTS TAKEN INTO ACCOUNT

PMM Vol.41, № 1, 1977, pp.34-40

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(Received June 15, 1976)

A variational equation is used within the framework of the special theory of relativity, to derive a system of equations of mechanics and electrodynamics describing the behavior of a polarizable and magnetizable continuous medium. General dynamic equations are used to derive and assess the relativistic equations of impulses for a continuous medium, for a number of concrete models of continuous media.

Let x^1, x^2, x^3 and $x^4 = ct$ denote the coordinates in an arbitrarily chosen inertial frame of reference of the observer with the metric signature $(- - - +)$, and c the speed of light in vacuum; let also ξ^1, ξ^2, ξ^3 and $\xi^4 = c\tau$ be the coordinates of the points of the medium in the moving concomitant coordinate system frozen into the medium and $x^i = x^i(\xi^k)$ be the law of motion of the medium. We denote by g_{ij} and $\hat{g}_{ij}$ the components of the metric tensor in the observer's system and in the comoving coordinate system respectively, $g_{ij}dx^i dx^j = \hat{g}_{ij}d\xi^i d\xi^j = ds^2$; by $\rho = \rho_0(\xi^\mu)[\det \| g_{ij}\hat{} - u_i u_i\hat{} \|]^{-1/2}$ the mass density of the medium; by $u^i = \partial x^i / \partial \xi^4$ the contravariant components of the four-dimensional velocity vector and by x_j^i the derivative $\partial x^i / \partial \xi^j$. (Here and henceforth the lower case Italics indices assume the values 1, 2, 3, 4 and the Greek letter indices the values 1, 2, 3). To describe the effects of the interaction between the electromagnetic field and the polarizable and magnetizable medium, we introduce the following antisymmetric electromagnetic field tensors with the components F_{ij} and H^{ij} :

$$F_{ij} = \begin{Vmatrix} 0 & B^3 & -B^2 & E_1 \\ -B^3 & 0 & B^1 & E_2 \\ B^2 & -B^1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{Vmatrix}, \quad H^{ij} = \begin{Vmatrix} 0 & H_3 & -H_2 & -D^1 \\ -H_3 & 0 & H_1 & -D^2 \\ H_2 & -H_1 & 0 & -D^3 \\ D^1 & D^2 & D^3 & 0 \end{Vmatrix}$$

To obtain the defining equations of electrodynamics and mechanics of continuous media, we use the variational equation in the form [1]

$$\delta \int_{V_4} \Lambda \, d\tau_4 + \delta W^* + \delta W = 0 \tag{1}$$

Here $d\tau_4$ denotes a four-dimensional element of an arbitrary volume of the space of events V_4 bounded by a three-dimensional surface Σ_3. The volume V_4 may contain the surfaces of strong discontinuity S in the characteristics of the electromagnetic field and the medium, Λ is the Lagrange function and δW^* is a given functional which is introduced to account for the external forces acting on the medium — electromagnetic field system, and for the irreversible processes taking part within the medium. The functional δW is defined by specifying Λ and δW^*. The functions subjected to variation in the variational equation (1) are the unknown functions of ξ^k used to express Λ and δW^*. The unknown functions of ξ^k may be discontinuous, but their variations are, by definition, continuous.

Let us define the Lagrangian Λ in the following specific form:

$$\Lambda = \frac{1}{8\pi} F_{ik} H^{ik} + \frac{1}{4\pi} A_k \nabla_i H^{ik} - \frac{1}{4\pi} \nabla_i (A_k H^{ik}) + \Lambda_0 (x_j^i, F_{ik}, s, g_{ij}, K^{\text{B}}) \quad (2)$$

Here Λ_0 is the part of the Lagrangian whose dependence on its arguments may vary, depending on the properties of the medium, K^{B} denote the specified tensor components not subject to variation and characterizing the geometrical and physical properties of the medium, s is the specific entropy density and A_k denote the components of a vector to be specified later. The functional δW^* can be determined, within the framework of the phenomena under consideration, by the formula

$$\delta W^* = \int\limits_{V_4} [\delta Q^{(e)} - Q_i \delta x^i] \, d\tau_4 + \int\limits_{S} [F_i \delta x^i + \gamma^i \delta_L A_i] \, d\sigma_3 \quad (3)$$

(the choice of the sign preceding the term $Q_i \delta x^i$ is governed by the choice of the signature of the metric of the pseudo-Euclidean space $(- - - +)$, while the signature of the metric of the three-dimensional Euclidean subspace is $(+ + +)$) . In the formula (3) $\delta Q^{(e)}$ represents a virtual generalization of the heat influx process; $Q_i \delta x^i$ determines, for the real processes, the elementary work done by the external forces and the external flux of the nonthermal energy to the medium — electromagnetic field system; F_i are the components of the four-dimensional concentrated surface force; γ^i are the components of the four-dimensional surface electric current vector and $\delta_L A_i = \delta A_i + A_k \nabla_i \delta x^k$ is the absolute variation of the components of the vector A_i which, for the case of real increments, yields the increments in the vector components relative to the intrinsic coordinate system (i.e. in the inertial coordinate system attached to a particle of the medium and undergoing a translational motion relative to the observer's frame of reference at the instantaneous velocity of the medium particle).

In the case of equilibrium processes for which the absolute temperature T is specified, we use the virtual analogy of the second law of thermodynamics

$$\rho T \delta s = \delta Q^{(e)} + \delta Q' \quad (4)$$

where $\delta Q'$ denotes the generalized variation representing the analog of the uncompensated heat. We shall assume that in the case of a real process the possible increment in the amount of uncompensated heat is governed by the following three mechanisms only: the presence of the properties of viscosity in the medium, the heat conduction process and the emission of Joule heat. Consequently we can write the corresponding generalized expression in the form

$$\delta Q' = \tau_i^{\ j} \nabla_j \delta x^i - j^k \delta_L A_k \quad (5)$$

Here j^k are the components of the four-dimensional electric current vector, and the components of the four-dimensional tensor $\tau_i^{\ j}$ determine the viscous and heat conductivity properties of the medium.

Assuming now that $\delta A_k = \partial A_k + \delta x^i \nabla_i A_k$, $\delta H^{ik} = \partial H^{ik} + \delta x^j \nabla_j H^{ik}$, δs, $\delta F_{ik} = \partial F_{ik} + \delta x^j \nabla_j F_{ik}$ and δx^i are continuous and linearly independent, and using the relations (2) — (5), we obtain from (1), for the continuous processes, a system of equations of electrodynamics and mechanics as well as an expression for the functional δW ·

(here the variations of the functions are determined in accordance with [2, 3]).

The Maxwell equations are

$$\nabla_k H^{ik} = 4\pi j^i \quad \text{for } \partial A_i \tag{6}$$

$$F_{ik} = \nabla_i A_k - \nabla_k A_i \quad \text{и } \partial H^{ik}$$

the equations of state are

$$\rho T + \partial\Lambda_0 / \partial s = 0 \quad \text{при } \delta s \tag{7}$$

$$\frac{1}{8\pi} H^{ik} + \frac{\partial \Lambda_0}{\partial F_{ik}} = 0 \quad \text{for } \partial F_{ik}$$

and the impulse equations are

$$\nabla_k P_i^{\ k} = Q_i \quad \text{for } \delta x^i \tag{8}$$

$$\delta W = \int\limits_{\Sigma_3 + S_\pm} \left[P_i^{\ k} \delta x^i + \frac{1}{4\pi} H^{ki} \delta_L A_i \right] n_k d\sigma_3 \tag{9}$$

Here n_k are the components of the four-dimensional vector of the exterior normal to the surface $\Sigma_3 + S_\pm$ (the subscript $\pm$ means that the integration is carried out on both sides of the surface of discontinuity S), and the components of the energy — impulse tensor $P_i^{\ k}$ in (8) and (9) are determined from the relations which, together with (7), represent the equations of state

$$P_i^{\ k} = -\Lambda_0 \delta_i^{\ k} - \frac{\partial \Lambda_0}{\partial x_s^{\ i}} x_s^{\ k} - \frac{1}{4\pi} H^{kj} F_{ij} + \tau_i^{\ k} \tag{10}$$

From Eqs. (6) it follows that the vector $\mathbf{A}$ with components A_k represents a vector potential of the electromagnetic field. Using now Eqs. (6) — (8), the relations (10) with the equation of continuity $\nabla_i (\rho u^i) = 0$ for the medium taken into account and the definition of the derivative with respect to intrinsic time $u^i \nabla_i = c^{-1} (d / d\tau)$, we obtain the following scalar entropy balance equation:

$$\rho T \frac{ds}{d\tau} = c j^k F_{ki} u^i - c u^i \nabla_j \tau_i^{\ j} + \frac{\partial \Lambda_0}{\partial K^B} \frac{dK^B}{d\tau} + c \nabla_j \left(x_4^{\ i} x_s^{\ j} \frac{\partial \Lambda_0}{\partial x_s^{\ i}} \right) \tag{11}$$

Setting $i^k = j^k - \rho_e u^k$ where ρ_e is the electric charge density, we find that i^k represent the components of the electric conduction current vector. The four-dimensional invariant $c i^k F_{ki} u^i = c j^k F_{ki} u^i$ represents the Joule heat.

We note that within the framework of our model the momentum equations for the medium — electromagnetic field system follow from the impulse equations and the Maxwell equations. It can be seen that for the real processes $\Lambda = \Lambda_0$.

Assuming that the variations δx^i and $\delta_L A_i$ are continuous on S, we obtain from the variational equation the following relations on the surfaces of discontinuity:

$$[P_i^{\ k} n_k]_{S_\pm} + F_i = 0, \quad [H^{ki} n_k]_{S_\pm} + 4\pi \gamma^i = 0 \tag{12}$$

This represents the general theory for any Λ_0 which is determined by using the thermodynamic postulates referring to the nature of the energy of the medium and the field. In particular, Λ_0 can be given in the form

$$\Lambda_0 = -\frac{1}{16\pi} C_{ijkl} F^{ij} F^{kl} - U(x_j^{\ i}, s, g_{ij}, K^B) \tag{13}$$

where the following relations hold for the tensor components C_{ijkl}:

$$C_{ijkl} = C_{klij} = -C_{jikl} = -C_{ijlk}$$

and U is a scalar function of the thermodynamic parameters of the medium. The sense of the latter function will be explained below for certain particular cases. When the function Λ_0 is chosen in the form (13) under the assumption that $C_{ijkl} = C_{ijkl}(x_q{}^p, s, K^B)$, the equations of state for the electromagnetic field and the impulse equations for the medium — electromagnetic field system can be written in the form

$$H_{ij} = C_{ijkl}F^{kl} \tag{14}$$

$$\nabla_j\left[\frac{\partial U}{\partial x_s{}^i}x_s{}^j + U\delta_i{}^j + \tau_i{}^j + \frac{1}{16\pi}F^{mn}F^{pq}x_s{}^j\frac{\partial}{\partial x_s{}^i}C_{mnpq}\right] = Q_i + R_i \tag{15}$$

$$R_i = -\nabla_j S_i{}^j, \qquad S_i{}^j = -\frac{1}{4\pi}\left[H^{jk}F_{ik} - \frac{1}{4}F_{mn}H^{mn}\delta_i{}^j\right]$$

The case when the relation connecting the components of the tensors H_{ik} and F_{jl} is nonlinear, can also be described using the framework of the proposed model, if we assume that the arguments of the tensor function C_{ijkl} include the components of the tensor F_{mn}.

For a medium with isotropic electromagnetic properties the components of the tensor C_{ijkl} have the form [2]

$$C_{ijkl} = \frac{1}{2}\left[\frac{1}{\mu}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) + \varepsilon(g_{ik}u_j u_l - g_{jk}u_i u_l + g_{jl}u_i u_k - g_{il}u_j u_k)\right], \; \gamma_{ij} = g_{ij} - u_i u_j \tag{16}$$

where ε and μ are the coefficients of the dielectric and magnetic permeability of the medium. In the present model these coefficients can, generally speaking, depend on the entropy s and the invariants of the tensors $x_j{}^i$ and K^B.

Below we consider the case in which the components of the tensor C_{ijkl} are given by the formulas (16), $\varepsilon = \varepsilon(\rho, s)$ and $\mu = \mu(\rho, s)$. Then we can transform Eqs.(15) which are valid in any inertial frame of reference, to the form

$$\nabla_j\left[\frac{\partial U}{\partial x_s{}^i}x_s{}^j + U\delta_i{}^j + \tau_i{}^j\right] = -\frac{1}{8\pi}\nabla_j\left\{\left[\frac{1}{2}\frac{\rho}{\mu^2}\frac{\partial\mu}{\partial\rho}F_{mn}F^{mn} - \rho\left(\frac{1}{\mu^2}\frac{\partial\mu}{\partial\rho} + \frac{\partial\varepsilon}{\partial\rho}\right)F^{mn}F^{pq}g_{mp}u_n u_q\right]\gamma_i{}^j\right\} + Q_i + R_i - \frac{\rho}{4\pi c}\frac{d}{d\tau}\left[\frac{\varepsilon\mu - 1}{\mu\rho}F^{mn}F^{lq}g_{nq}u_m\gamma_{li}\right] \tag{17}$$

The system of equations (6),(7),(14) and (17) defines, in particular, a model of an elastic body with possible anisotropic mechanical characteristics. Further concretization of the model involves specifying the form of the scalars U, ε and μ, and of the components of the tensors $\tau_i{}^j$ and i^k as functions of the defining parameters of the model.

As a particular case we consider a relativistic model of an ideal compressible isotropic fluid, assuming that the external energy flux to the medium — electromagnetic field system is absent. If we also assume that the scalar function U depends on the arguments ρ, u^i, s and g_{ij} and represents the energy of the medium calculated per unit vloume, we can write by definition

$$U = \rho c^2 + \rho U_0(\rho, s)$$

where U_0 is a function of mass density of the additional internal energy of the medium.

After the transformation the impulse equations (17) become

$$\rho \frac{d}{d\tau}\left[cu_i + \frac{1}{c} U_0 u_i\right] = -\nabla_j (p\gamma_i^{\,j}) + Q_i + R_i + N_i \tag{18}$$

$$p = \frac{1}{16\pi}\frac{\rho}{\mu^2}\frac{\partial\mu}{\partial\rho}F^{mn}F_{mn} - \frac{\rho}{8\pi}\left[\frac{1}{\mu^2}\frac{\partial\mu}{\partial\rho} + \frac{\partial\varepsilon}{\partial\rho}\right]F^{mn}F^{pq}g_{mp}u_n u_q + \rho^2\frac{\partial U_0}{\partial\rho}$$

$$N_i = -\frac{\rho}{4\pi c}\frac{d}{d\tau}\left[\frac{\varepsilon\mu - 1}{\mu\rho}F^{kj}F^{sl}g_{jl}u_k\gamma_{si}\right]$$

where p denotes, by definition, the total pressure in the fluid.

In the absence of an electromagnetic field, the impulse equations (18) coincide with the equations of the special theory of relativity.

Let us consider in greater detail the right-hand side of the four-dimensional impulse equations (18).

In the intrinsic coordinate system the total pressure in the fluid can be written in the form

$$p = -\frac{\rho}{8\pi}\left(\frac{\partial\mu}{\partial\rho}H^2 + \frac{\partial\varepsilon}{\partial\rho}E^2\right) + \rho^2\frac{\partial U_0}{\partial\rho}$$

and this is often interpreted as the sum of the hydrodynamic, electrostriction and magnetostriction pressures.

Using the three-dimensional vector characteristics in the intrinsic coordinate system to describe the electromagnetic field in the medium, we can write the first three components of the four-dimensional vector N_i appearing in the right-hand side of (18), for the case of a fluid at rest, in the form

$$N_\alpha = \frac{\varepsilon\mu - 1}{4\pi c}\frac{d}{d\tau}\left[\varepsilon_{\alpha\beta\gamma}E^\beta H^\gamma\right]$$

where $\varepsilon_{\alpha\beta\gamma}$ are the components of the completely antisymmetric three-dimensional Levi-Civita tensor.

Considering that for a fluid at rest we have $d / d\tau = \partial / \partial\tau$, and using the fact that the metrics of the four dimensional pseudo-Euclidean space and of the three-dimensional Euclidean subspace have different signatures, we can show that the sum $R_\alpha + N_\alpha$ in the nonrelativistic approximation coincides with the corresponding expression [4] for the components of the volume ponderomotive force vector acting on the medium from the direction of the electromagnetic field.

It should be pointed out that the fourth component of the vector N_i appearing in the right-hand side of (18) is, in the case of a relativistic fluid, usually different from zero. We note that the components of the four-dimensional vector N_i which depend essentially on the particular properties of the fluid through the parameters ε and μ, can be combined with the left-hand side of (18). Subsequently, the left-hand side of (18) can be interpreted as the change in the four-dimensional impulse vector of the medium per unit intrinsic time, made more complicated by inclusion of polarization and magnetization. In this case the vector with components R_i has the meaning of the four-dimensional volume ponderomotive force vector calculated in accordance with the form of the energy—impulse tensor of the Minkowski electromagnetic field. Clearly, the form of (18) is independent of the above discrepancy in the interpretation of the medium impulse.

Passing in (18) to the limit, we obtain the nonrelativistic equations of the motion and the energy equation for a polarizable and magnetizable fluid. We note that the simple formulas and conclusions obtained above depend on the important assumptions which were made about the form of the Lagrangian Λ_0 and on the assumption that the medium is mechanically and electromagnetically isotropic.

In the case of anisotropic media the generalized formulas (16) can easily be obtained by the general methods [5]. After this we can use the four-dimensional impulse equations in the form (15) to obtain for the media, which are electrodynamically anisotropic, the exact relativistic impulse equations describing the motions of anisotropic media within the framework of the special theory of relativity.

The system of equations derived here is equivalent to the equations obtained in [3, 6]; the approach used here has, however, a number of advantages. If the medium cannot be polarized and magnetized, then the system splits into the subsystems of equations of mechanics and electrodynamics. (To achieve this it is sufficient to set $\varepsilon = \mu = 1$, while in the earlier equations [3] such an approach led to certain complications). Setting $\varepsilon = 1$ and $\mu \neq 1$, we obtain a model of a medium which can only be magnetized, while setting $\varepsilon \neq 1$ and $\mu = 1$ yields a system of equations describing a medium which can only be polarized.

We note that if the components of the electromagnetic field tensors F_{ij} and H_{ij} are linearly connected by relations of the type (14), then by virtue of the definition of the polarization — magnetization tensor $M_{ij} = (1/4\pi)(F_{ij} - H_{ij})$ the components of the tensors F_{ij} and M_{ij} will also be linearly dependent on each other.

The approach used here can easily be extended to embrace the case when Λ_0 depends on the gradients of the tensor F_{ij}. In this case the second equation of (7) in the system (6) — (9) becomes

$$\frac{1}{8\pi} H^{ik} + \frac{\partial \Lambda_0}{\partial F_{ik}} - \nabla_s \left(\frac{\partial \Lambda_0}{\partial \nabla_s F_{ik}} \right) = 0$$

Moreover the expressions for the complements of the tensor P_i^k and the functional δW become more complicated and additional relations at the surface of discontinuity S [2] are obtained to supplement the system (12).

In conclusion the author wishes to express his deep gratitude to L. I. Sedov for detailed assessment of this paper.

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Translated by L. K.